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CONTACT BETWEEN PLATES BY A NEW DIRECT BOUNDARY INTEGRAL EQUATION FORMULATION

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Abstract—The unilateral contact between two plates according to Kirchhoff's theory is solved by using a boundary integral equation method. A discretization leading to a matrix formulation is proposed. To solve this problem an elimination of boundary unknowns is performed. An iterative method proves successful with the example treated.

1. INTRODUCTION

For bodies in contact the contact surface is a very important unknown. With regards to the plate contact, Weitsman [1], Pu and Hussain [2] or Gladwell and Iyer [3] have presented methods limited to axisymmetric problems. In these cases, solutions can be obtained in terms of Fredholm integral or in terms of functional to be minimized. In recent years, Kartvelishvili [4] used a variational method in which the potential energy system is the minimized functional. The direct minimization is performed by the finite difference method.

In this paper we develop an original boundary integral equation method in which we eliminate the boundary unknowns to solve the contact problem between two plates. As the existence and the unicity of the solution have been proved by D. Fortuné [5, 6], we can establish an iterative procedure on the boundary contact domain. The examples solved prove that our original method is very efficient for this class of problems.

2. GOVERNING EQUATIONS

Consider two thin plates (Fig. 1):

—The first one (domain S_1, boundary Γ_1) has his mid-surface in the plane $z = 0$. The normal deflection at a point $P_1 \in S_1$ of coordinates x_1, y_1 and $z_1 = 0$ in the basis (0, x, y, z) is denoted by w_1.

—The second one (domain S_2, boundary Γ_2) has his mid-surface in the plane $z = -\delta$. The normal deflection at a point $P_2 \in S_2$ of coordinates x_2, y_2 and $z_2 = -\delta$ in the same basis (0, x, y, z) is denoted by w_2.

Consequently, according to Kirchhoff's theory of thin plate flexure, the transverse

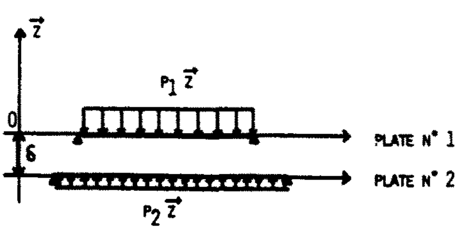

Fig. 1. Plates position.

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deflections w_1 and w_2 are governed by the two differential equations:

$$\Delta\Delta w_1 = \frac{P_1}{D_1} \text{ in } S_1 \tag{1}$$

$$\Delta\Delta w_2 = \frac{P_2}{D_2} \text{ in } S_2 \tag{2}$$

where Δ is the Laplacian $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, D_1 and D_2 are the flexural rigidities of plates (1) and (2), P_1 and P_2 are the loads per unit area on the plates (1) and (2), respectively.

Boundary conditions are associated with each differential equation. They bear upon two of the following quantities: the deflection w_1, its normal derivative $(\partial w_1/\partial n)$, the normal bending moment $M_n(w_1)$ and the equivalent transverse shear force $K_n(w_1)$ (respectively w_2, $(\partial w_2/\partial n)$, $M_n(w_2)$ and $K_n(w_2)$) where $\mathbf{n}$ is the outward unit normal on the boundary.

Suppose that the plate No. 1 is "above" the plate No. 2. The assumption of contact domain S_c, leads to relative contact loads, P_r acting upon the plate No. 1 and $-P_r$ upon the plate No. 2.

Therefore, if S_c is the contact domain, we have to solve the two simultaneous differential equations:

$$\Delta\Delta w_1 = \frac{P_1 + P_r}{D_1} \text{ in } S_1 \tag{3}$$

$$\Delta\Delta w_2 = \frac{P_2 - P_r}{D_2} \text{ in } S_2 \tag{4}$$

so that:

$$P_r > 0 \text{ and } w_1 - w_2 = -\delta \quad \text{in the contact domain } S_c \tag{5}$$

$$P_r = 0 \text{ and } w_1 - w_2 > -\delta \quad \text{out of } S_c. \tag{6}$$

Each of these equations is associated with two suitable Kirchhoff boundary conditions.

In this work we limit our study to the slightly simpler problem of two similar plates. In this case we have:

$$S_1 \equiv S_2,\ \Gamma_1 \equiv \Gamma_2 \text{ and } D_1 = D_2. \tag{7}$$

If boundary conditions on Γ_1 and Γ_2 are absolutely identical, we can limit our study to the following problem.

$$\Delta\Delta w = \frac{P_1 - P_2 + 2P_r}{D} \text{ in } S \equiv S_1 \equiv S_2 \tag{8}$$

with

$$D = D_1 = D_2,\ w = w_1 - w_2 \tag{9}$$

$$P_r > 0 \quad \text{and} \quad w = -\delta \quad \text{in } S_c \tag{10}$$

$$P_r = 0 \quad \text{and} \quad w > -\delta \quad \text{out of } S_c. \tag{11}$$

Note that this new problem is similar to the contact problem between a plate and a rigid ground initially distant from δ.

3. BOUNDARY ELEMENT METHOD

The formulation of static flexure plate problems by boundary integral equations is now possible. One [7, 8] of the most convenient approaches consists in taking the Rayleigh Green identity as the reciprocal theorem, and it leads to the two following integral equations:

$$\beta w(P) = \frac{1}{D}\int_S v(P,Q)p(Q)\,\mathrm{d}S_Q - \frac{1}{D}\int_\Gamma [K_n(v(P,Q))w(Q) - M_n(v(P,Q)) \times \frac{\partial w}{\partial n}(Q) + \frac{\partial v}{\partial n}(P,Q)M_n(w(Q)) - v(P,Q)K_n(w(Q))]\,\mathrm{d}s_Q - \frac{1}{D}\sum_{i=1}^{N} [\![w(A_i)M_{nt}(v(P,A_i)) - M_{nt}(w(A_i))v(P,A_i)]\!]_{A_i} \tag{12}$$

with

$$\beta = 1 \text{ if } P \in S \tag{13}$$

$$\beta = \frac{1}{2} \text{ if } P \in \Gamma \tag{14}$$

and

$$\frac{1}{2}\frac{\partial w}{\partial n_p}(P) = \frac{1}{D}\int_S \frac{\partial v}{\partial n_p}(P,Q)p(Q)\,\mathrm{d}S_Q - \frac{1}{D}\int_\Gamma \left[\frac{\partial K_n}{\partial n_p}(v(P,Q))\,w(Q) - \frac{\partial M_n}{\partial n_p}(v(P,Q))\frac{\partial w}{\partial n}(Q) + \frac{\partial^2 v}{\partial n_p \partial n}(P,Q)\,M_n(w(Q)) - \frac{\partial v}{\partial n_p}(P,Q)K_n(w(Q))\right]\mathrm{d}s_Q - \frac{1}{D}\sum_{i=1}^{N}\left[\!\!\left[w(A_i)\frac{\partial M_{nt}}{\partial n_p}(v(P,A_i)) - M_{nt}(w(A_i))\frac{\partial v}{\partial n_p}(P,A_i)\right]\!\!\right]_{A_i} \tag{15}$$

with

$$P \in \Gamma$$

where Q is a point on the boundary Γ; $\mathrm{d}S_Q$ (or $\mathrm{d}s_Q$) denotes the integration element on S (or Γ) with respect to the co-ordinates of Q; n is the outward unit normal at the point Q of the boundary Γ; n_p is the outward unit normal at the point P of the boundary Γ; $K_n(u)$ is the Kirchhoff transverse shear force associated with the deflection field u; $M_n(u)$ is the normal flexure moment associated with the deflection field u; $M_{nt}(u)$ is the torsion moment associated with the deflection field u; $v(P,Q)$ is the fundamental solution of $\Delta\Delta v = 0$, whose second and third derivatives exhibit a singularity at P, such that $v(P,Q) = (1/8\pi)r^2 \log r$ where $r = \| PQ \|$; $[\![\cdot]\!]_{A_i}$ is the jump of the function which may occur at N corners A_i of curvilinear abscissa s_i, defined by $[\![\cdot]\!]_{A_i} = (\cdot)s_i^+ - (\cdot)s_i^-$; $p(Q)$ is the transverse load at the point Q of the domain S; and N is the number of corners of the plate edge.

Along the boundary, the known quantities are: $K_n(w)$ and $M_n(w)$ on a free edge; w and $M_n(w)$ on a simply supported edge; w and $\partial w/\partial n$ on a clamped edge.

Furthermore, the quantity $M_{nt}(w)$ at a point Q can be expressed in terms of $\partial w/\partial n$:

$$M_{nt}(w(Q)) = -D(1-\nu)\frac{\partial}{\partial s}\left[\frac{\partial w}{\partial n}(Q)\right]. \tag{16}$$

4. MATRIX FORMULATION FOR PLATE WITH CONDITIONS INSIDE THE DOMAIN

Using the approach proposed for static problems in [9], a matrix formulation of contact problem can be performed:

—by a discretization of the boundary into q straight elements at the middle (nodal points) of which we define the value of deflection w, its normal derivative $(\partial w/\partial n)$, bending moment $M_n(w)$ and transverse shear $K_n(w)$.

—by a discretization of the domain in k rectangular panels at the middle (nodal points) of which we define the value of the deflection and the transverse load $(P_1 - P_2 + 2P_r)$ (Fig. 2).

When $M_{nr}(w)$ is expressed in terms of $(\partial w/\partial n)$, the two equations (12) together with (14) and (15) can be discretized in the following way:

$$\frac{1}{2}\{w\} = [A_\Gamma]\{K_n\} + [B_\Gamma]\{M_n\} + [C_\Gamma]\left\{\frac{\partial w}{\partial n}\right\} + [D_\Gamma]\{w\} + [E_\Gamma]\{F\} \tag{17}$$

$$\frac{1}{2}\left\{\frac{\partial w}{\partial n}\right\} = [A'_\Gamma]\{K_n\} + [B'_\Gamma]\{M_n\} + [C'_\Gamma]\left\{\frac{\partial w}{\partial n}\right\} + [D'_\Gamma]\{w\} + [E'_\Gamma]\{F\} \tag{18}$$

where $[A_\Gamma]$, $[B_\Gamma]$, $[C_\Gamma]$, $[D_\Gamma]$ and $[A'_\Gamma]$, $[B'_\Gamma]$, $[C'_\Gamma]$, $[D'_\Gamma]$ are q by q matrix whose coefficients result from the curvilinear integrals of (12) and (15), respectively; $[E_\Gamma]$ and $[E'_\Gamma]$ are q by k matrix whose coefficients arise from the surface integrals of (12) and (15), respectively; $\{w\}$, $\{\partial w/\partial n\}$, $\{M_n\}$ and $\{K_n\}$ are the column vectors of quantities defined on the q nodal points of Γ; and $\{F\}$ is the column vector whose k components are the transverse load per unit area at the k nodal points inside the domain S.

In the case of homogeneous boundary conditions, eqns (17) and (18) can be condensed in the following way:

$$[G_\Gamma]\{I\} + [J_\Gamma]\{F\} = \{0\} \tag{19}$$

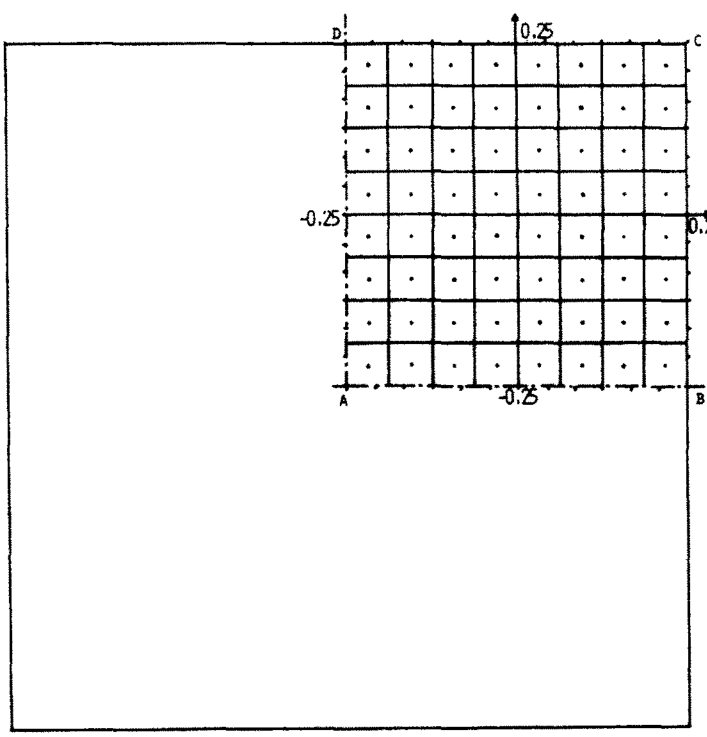

Fig. 2. Boundary and domain discretization.

where $[G_\Gamma]$ is a $2q$ by $2q$ matrix deduced from $[A_\Gamma]$, $[B_\Gamma]$, $[C_\Gamma]$, $[D_\Gamma]$, $[A'_\Gamma]$, $[B'_\Gamma]$, $[C'_\Gamma]$ and $[D'_\Gamma]$; $[J_\Gamma]$ is a $2q$ by k matrix deduced from $[E_\Gamma]$ and $[E'_\Gamma]$; $\{I\}$ is the vector whose $2q$ components are the $2q$ boundary unknowns (among w, $(\partial w/\partial n)$, M_n and K_n).

In the same way, eqn (12) together with (13) written for each nodal point P inside the domain S gives:

$$\{w_s\} = [G_s]\{I\} + [J_s]\{F\} \tag{20}$$

where $[G_s]$ is a k by $2q$ rectangular matrix; $[J_s]$ is a k by k matrix; and $\{w_s\}$ is the vector whose k components are the k values taken by the deflection at the k nodal points inside the domain S.

A very convenient method [10], as we prove latter, consists in eliminating the unknown vector $\{I\}$ in (20) by inverting eqn (19) in the form:

$$\{I\} = -[G_\Gamma^{-1}][J_\Gamma]\{F\} \tag{21}$$

where $[G_\Gamma^{-1}]$ is the inverse matrix of $[G_\Gamma]$.

By substituting (21) into (20) we obtain:

$$\{w_s\} = (-[G_s][G_\Gamma^{-1}][J_\Gamma] + [J_s])\{F\}. \tag{22}$$

In the contact problem the unknowns are: w_s for a point P out of S_c, in this case the corresponding component of $\{F\}$ is $(P_1 - P_2)$; F for a point P in S_c, in this case the corresponding component of $\{w_s\}$ is $-\delta$.

Consequently for a given domain S_c, (12) is a linear system of k unknowns, which can be recast in the form:

$$\begin{Bmatrix} w_{s_1} \\ w_{s_2} \end{Bmatrix} = (-[G_s][G_\Gamma^{-1}][J_\Gamma] + [J_s]) \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{23}$$

where $\{F_1\}$ is the known part of $\{F\}$ (so that $P_r = 0$) and $\{F_2\}$ is the unknown part; $\{w_{s_1}\}$ is the unknown part of $\{w_s\}$ and $\{w_{s_2}\}$ the known part so that $(w_s = -\delta)$.

This problem would be classical if the contact zone S_c were defined, but it is also an unknown.

Then, the effective problem becomes: what is the contact zone S_c?

5. DETERMINATION OF CONTACT ZONE

In way to find the contact zone we use an iterative procedure as follows:

First step

We suppose that there is no contact between the two plates. In this case we compute the deflection w in all the nodal points inside the domain for which the condition (11) is satisfied, consequently the given load F_1 is $(P_1 - P_2)$ inside S. The eqn (23) becomes:

$$\{w_{s_1}\} = (-[G_s][G_\Gamma^{-1}][J_\Gamma] + [J_s])\{F_1\}. \tag{24}$$

Second step

From last step, we consider whether the condition $w > -\delta$ is satisfied everywhere. If, for some points this condition is not satisfied, one plate has penetrated the other. For these nodes, we shall prescribe the contact condition $w = -\delta$, indeed $\{w_{s_2}\} = \{-\delta\}$ and the reaction P_r therefore the load vector $\{F_2\}$ will be unknown. The computation of $\{w_{s_1}\}$ and $\{F_2\}$ allows us to approach the following step.

Third step

If, among the nodes where we have prescribed $w = -\delta$ we have computed a negative reaction load $(P_r < 0)$ which is inconsistent with the contact condition, we shall prescribe

$P_r = 0$ for these points and for the others we keep the condition $w = -\delta$. We make a new computation with this assumption.

Fourth step

We repeat this procedure (steps two and three) until the contact zone no longer changes. When the problem set by relations (8)–(11) is completely solved, S_c and P_r are known. Consequently it is easy to compute w_1 and w_2 for plates (1) and (2), by solving (3), (4) and (23).

6. NUMERICAL RESULTS

We studied the problem of two identical square plates simply supported along the boundary, consequently:

$$w = 0$$

$$M_n(w) = 0.$$

The Poisson ratio is taken to be 0.3 and we give all the results for dimensionless variables x/a and y/a. In these conditions the loads per unit area are (Fig. 3):

—for the plate (1): $P_1 = 1$
—for the plate (2): $P_2 = 0$,

for a given distance between the two plates

$$\delta = 0{,}125.10^{-3}\frac{P_1 a^4}{D}.$$

Owing to symmetry we only study a quarter plate (Fig. 3); the half side is divided into 12 segments where the four quantities w, $(\partial w/\partial n)$, M_n and K_n are supposed to be constant. The interior domain is divided into 64 (8 × 8) square panels with the deflection assumed to be constant over each panel.

Boundary conditions are

$$w = 0 \quad \text{and} \quad M_n(w) = 0 \quad \text{on} \quad BC \text{ and } CD$$

and, by symmetry

$$\frac{\partial w}{\partial n} = 0 \quad \text{and} \quad K_n(w) = 0 \qquad \text{on} \quad AB \text{ and } DA$$

On Figs. 4 and 5 we have plotted the deflection along a symmetry axis and along a diagonal,

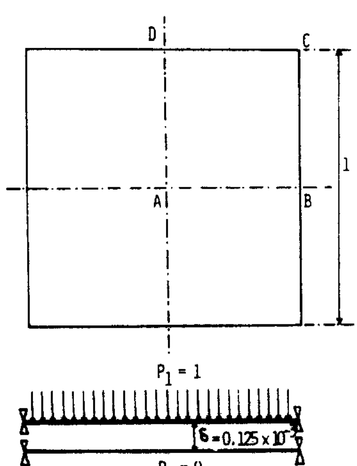

Fig. 3. Simply-supported plates.

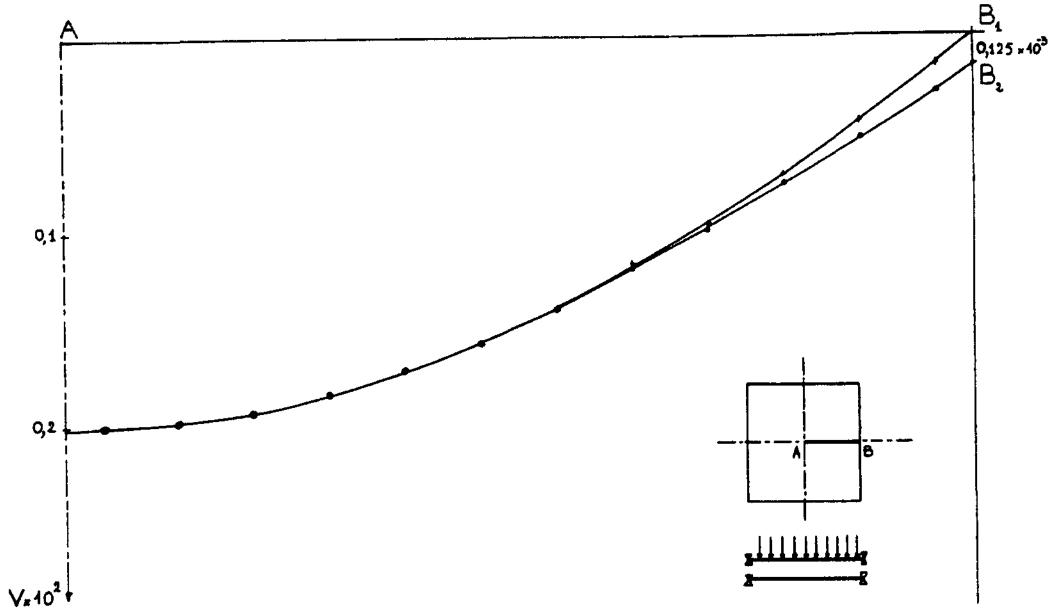

Fig. 4. Deflections w along AB after 19 iterations.

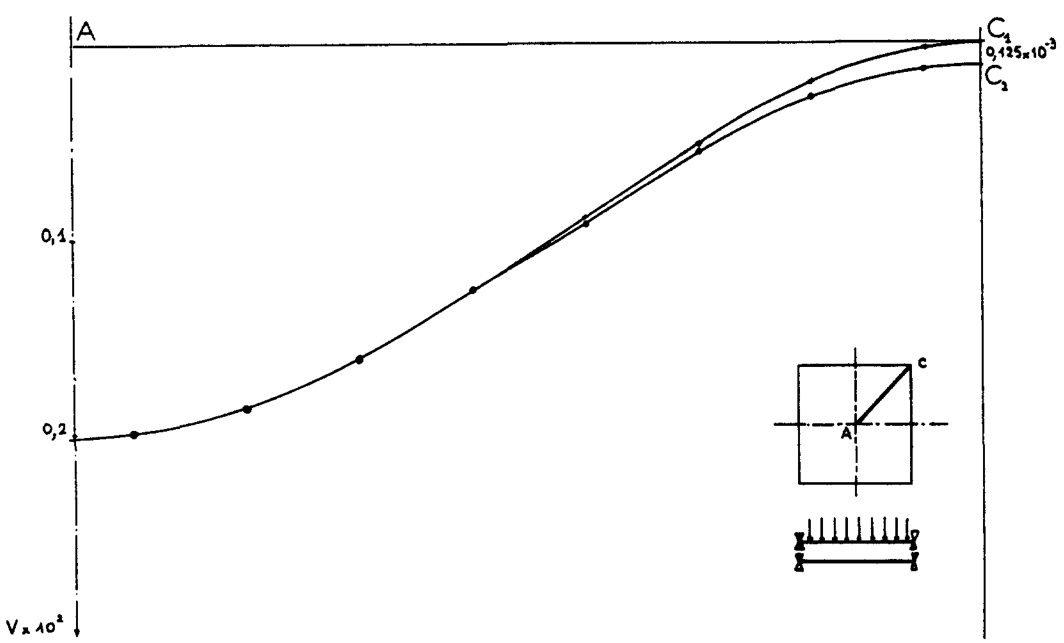

Fig. 5. Deflections w along AC after 19 iterations.

respectively, of the plates. We also displayed the contact zone on Fig. 6. These values are obtained in 19 iterations after which we found stability in the numerical results.

As comparison element we noted that the maximal deflection computed, namely $w = 3{,}91\ (P_1a^4/D)$ differs by less than 4% from the maximal deflection calculated from the exact solution [11].

7. CONCLUSION

In the paper the modified boundary integral method is presented and illustrated by solving the contact problem between two rectangular plates. The classical method [7] by boundary integral equations, needs to solve simultaneously:

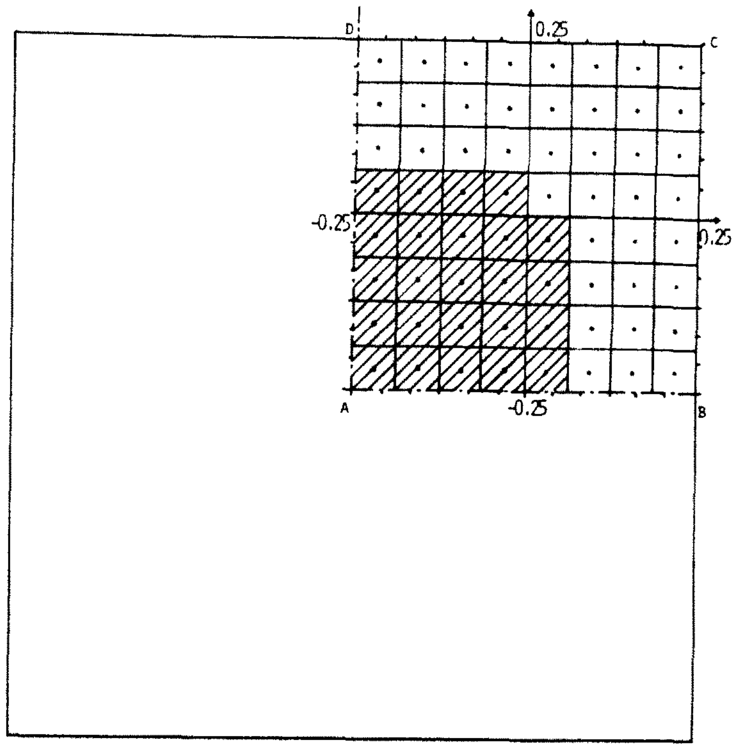

Fig. 6. Contact domain after 19 iterations.

- 96 equations for the boundary
- 64 equations for the domain

for an iterative calculus.

Consequently for solving this problem, the classical method leads to 19 systems of 160 equations.

Our method leads to 19 systems of 64 equations. The gain of computational time is very importatnt. However, this method has the usual disadvantages related to matrix inversions. In order to keep the numerical work within reasonable extend, the number of boundary elements has to be suitably restricted.

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